THE SCOTS COLLEGE

Year 12 Extension II Mathematics
ASSESSMENT TASK 1

Friday 17th February 2012
Format: 45 minute examination

Weighting of Task: 10%

Instructions:

- **Do not open this paper until instructed to do so.**
- Show ALL necessary working to gain full marks.
- Board Approved calculators may be used.
- Diagrams are NOT to scale.
- Marks may be deducted for careless or incomplete working.
- **Answer the free response questions (Section A) on the multiple choice answer sheet provided.**
- **Answer Sections B and C on the separate answer sheets provided.**
- **Start each question on a new page.**
- Attempt **all** questions

Section	Topics	Marks
A	Free Response questions	/5
B	Complex Numbers Graphs	/20 /15
	TOTAL	/40

Section A – MULTIPLE CHOICE QUESTIONS (5 Marks)
Each question is worth **1 mark**.
Circle the correct response on the answer sheet paper.

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1. In $x + iy$ form, $(2-3i)^2$ is equivalent to:

(A) $-5-12i$

(B) $-5+12i$

(C) $13-12i$

(D) $13+12i$

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2. The graph of the function with rule $f(x) = 2x - \dfrac{1}{x^2}$ has:

(A) one asymptote and a local minimum at $(-1, -3)$

(B) one asymptote and a local maximum at $(-1, -3)$

(C) two asymptotes and a local minimum at $(-1, -3)$

(D) two asymptotes and a local maximum at $(-1, -3)$

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3. The distance between the two points z and $-\overline{z}$ in the complex plane is given by:

(A) $2\,|\mathrm{Re}\,(z)|$

(B) $2\,|\mathrm{Im}\,(z)|$

(C) $2|z|$

(D) $2\,|\mathrm{Re}\,(z)| + 2\,|\mathrm{Im}\,(z)|$

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4. The slope of the curve $2x^3 - y^2 = 7$ at the point (2,3) is given by:

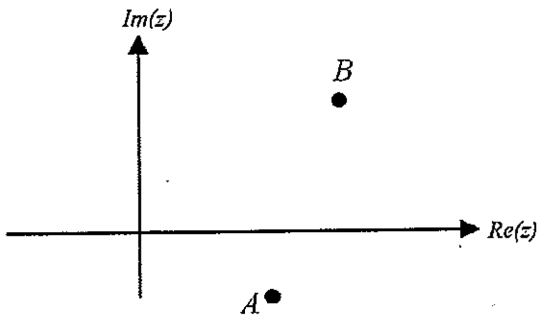
(A) -4

(B) -2

(C) 2

(D) 4

5.



The diagram above shows the points A and B representing the complex numbers $4-2i$ and $6+4i$ respectively. The locus of the circle with diameter AB is given by the equation $|z-k|=r$, where k is a complex number and r is a positive real number. Which of the following statements is true?

(A) $k=-5-i$ and $r=\sqrt{10}$

(B) $k=5+i$ and $r=\sqrt{10}$

(C) $k=-5-i$ and $r=10$

(E) $k=5+i$ and $r=10$

END OF SECTION A

Section B - EXTENDED RESPONSES (35 Marks)

- Show ALL necessary working to gain full marks.
- ***Write your responses on writing paper supplied.***
- Clearly write your name and class teacher on each page to be marked.
- Begin each question on a new page.

Section B - QUESTION 1: Complex Numbers (20 Marks)

(a) If $z = (1+\sqrt{3}i)(1+i)$,

(i) Express z in the form $x+iy$ where x and y are real numbers — 2

(ii) Express both $(1+\sqrt{3}i)$ and $(1+i)$ in mod-arg form and then show that $z = \sqrt{8}\,cis\left(\frac{7\pi}{12}\right)$ — 2

(iii) Hence find the exact values of $\cos\frac{5\pi}{12}$ and $\sin\frac{5\pi}{12}$ — 2

(iv) What is the smallest positive integer n for which z^n is real — 1

(b) Let $z = cis\frac{\pi}{4}$ — 3

On the **same** Argand plane, plot **and** label carefully **all** the points:

$z, \bar{z}, -2z, iz, z^2, 2z$

(c) (i) Express $-2-2\sqrt{3}i$ in modulus-argument form. — 2

(ii) Evaluate $\left(-2-2\sqrt{3}i\right)^6$ using De Moivrés theorem. — 2

Section B - QUESTION 1: (continued)

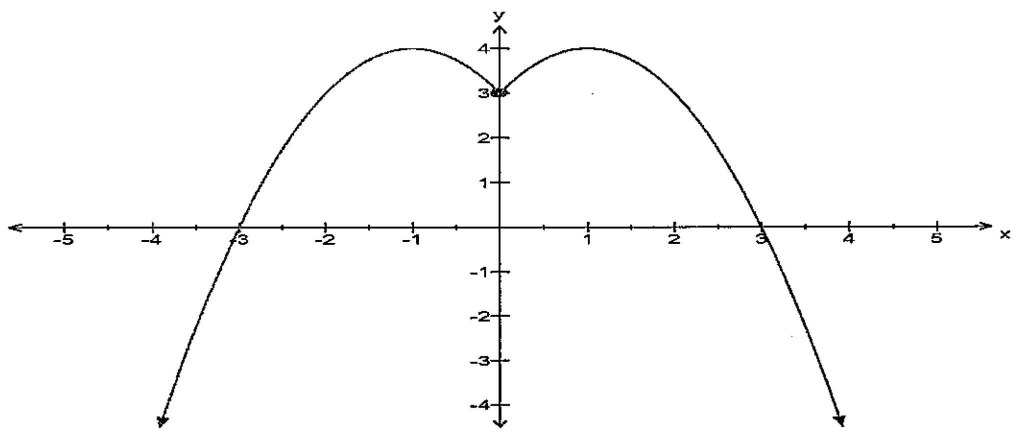
(d) If ω is a complex root of unity and using the fact that $1+\omega+\omega^2 = 0$, evaluate $(1-3\omega+\omega^2)(1+\omega-8\omega^2)$ — 3

(e) On an Argand diagram, neatly shade the region that holds simultaneously for both inequalities: — 3

$|z-(2+i)| \leq \sqrt{5}$ and $Arg\ z < \dfrac{\pi}{12}$

Section B - QUESTION 2: Graphs (15 Marks) Start a new answer page

(a) The sketch below is of the even function $y=f(x)$.



On separate number planes sketch each of the following, clearly showing all important features.

(i) $y=f(x)-2$ 1

(ii) $y=f(x-2)$ 1

(iii) $y=|f(x)|$ 1

(iv) $y^2=f(x)$ 2

(v) $y=\dfrac{1}{f(x)}$ 2

(b) Consider the function $f(x) = \dfrac{e^x - 1}{e^x + 1}$

(i) Show that the function is odd **1**

(ii) Show that the function is always increasing **2**

(iii) Find $f'(0)$ **2**

(iv) Sketch $y = f(x)$ showing any asymptotes **2**

(v) Use your graph to evaluate a set of values of M such that $\dfrac{e^x - 1}{e^x + 1} = Mx$ has 3 real solutions. **1**

END OF SECTION B
END OF EXAMINATION

Yr 12 Extension 2

Task 1 2012

Solutions.

Multi-choice

1. A
2. D
3. A
4. D
5. B

Section B

1. a) (i) $z = (1+\sqrt{3}i)(1+i)$

$= 1 + i + \sqrt{3}i - \sqrt{3}$ ✓

$= (1-\sqrt{3}) + i(1+\sqrt{3})$ ✓

(ii) $1+\sqrt{3}i = 2\operatorname{cis}\frac{\pi}{3}$ ✓

$1+i = \sqrt{2}\operatorname{cis}\frac{\pi}{4}$

$$\left(2\operatorname{cis}\frac{\pi}{3}\right)\left(\sqrt{2}\operatorname{cis}\frac{\pi}{4}\right) = 2\sqrt{2}\operatorname{cis}\left(\frac{\pi}{3}+\frac{\pi}{4}\right) \checkmark$$

$$= 2\sqrt{2}\operatorname{cis}\left(\frac{7\pi}{12}\right)$$

$$= \sqrt{8}\operatorname{cis}\left(\frac{7\pi}{12}\right)$$

(iii) $\sqrt{8}\cos\left(\frac{7\pi}{12}\right) = 1-\sqrt{3}$ and $\sqrt{8}\sin\left(\frac{7\pi}{12}\right) = 1+\sqrt{3}$.

$\therefore$ Equating real/imaginary parts.

$\therefore \cos\left(\frac{7\pi}{12}\right) = \frac{1-\sqrt{3}}{\sqrt{8}}$ $\qquad \sin\left(\frac{7\pi}{12}\right) = \frac{1+\sqrt{3}}{\sqrt{8}}$

$$\cos\left(\frac{5\pi}{12}\right) = \cos\left(\pi - \frac{7\pi}{12}\right)$$

$$= \cos\pi\cos\frac{7\pi}{12} + \sin\pi\sin\frac{7\pi}{12}$$

$$= -\left(\frac{1-\sqrt{3}}{\sqrt{8}}\right) \checkmark$$

or $\frac{\sqrt{3}-1}{\sqrt{8}}$

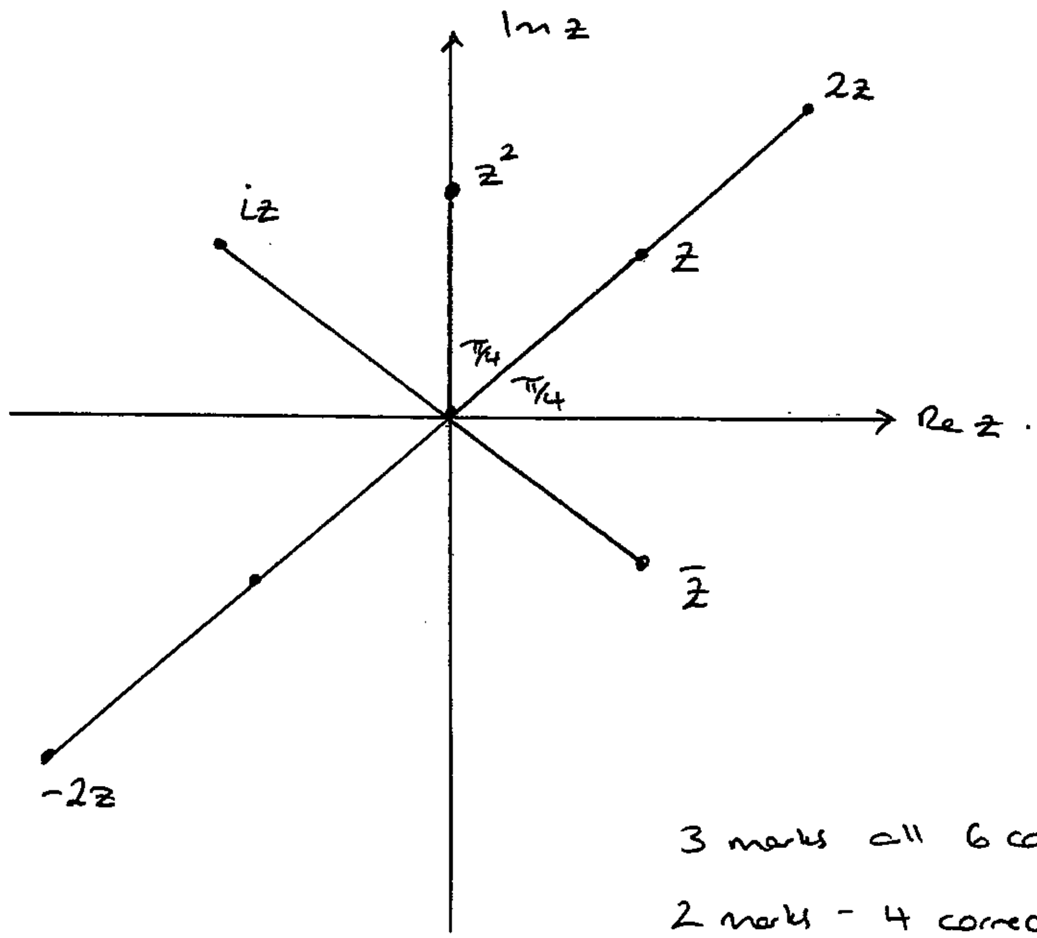
$$\sin\left(\frac{5\pi}{12}\right) = \sin\left(\pi - \frac{7\pi}{12}\right)$$
$$= \cancel{\sin\pi\cos\frac{7\pi}{12}} - \cos\pi \cdot \sin\frac{7\pi}{12}$$
$$= +\sin\left(\frac{7\pi}{12}\right)$$
$$= +\left(\frac{1+\sqrt{3}}{\sqrt{8}}\right).$$ ✓

(iv) Smallest integer of n:

$$z^n = \left[\sqrt{8}\operatorname{cis}\left(\frac{7\pi}{12}\right)\right]^n$$

in order for z^n to be real $\underline{n = 12}$. ✓

b)



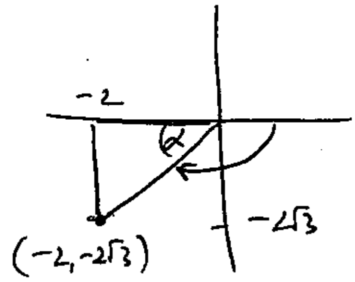
3 marks all 6 correct

2 marks - 4 correct

1 mark - 2 correct

c) (i) $-2-2\sqrt{3}i$

$= 4\,\text{cis}\left(-\frac{2\pi}{3}\right)$



modulus $= \sqrt{(-2)^2+(-2\sqrt{3})^2}$
$= 4$

Arg $= \pi - \tan^{-1}\left(\frac{b}{a}\right)$
$= -\frac{2\pi}{3}$

(ii) $(-2-2\sqrt{3}i)^6 = \left[4\,\text{cis}\left(-\frac{2\pi}{3}\right)\right]^6$

$= 4^6\,\text{cis}(-4\pi)$

$= 4^6$ (or 4096)

d) Using $1+\omega+\omega^2=0$

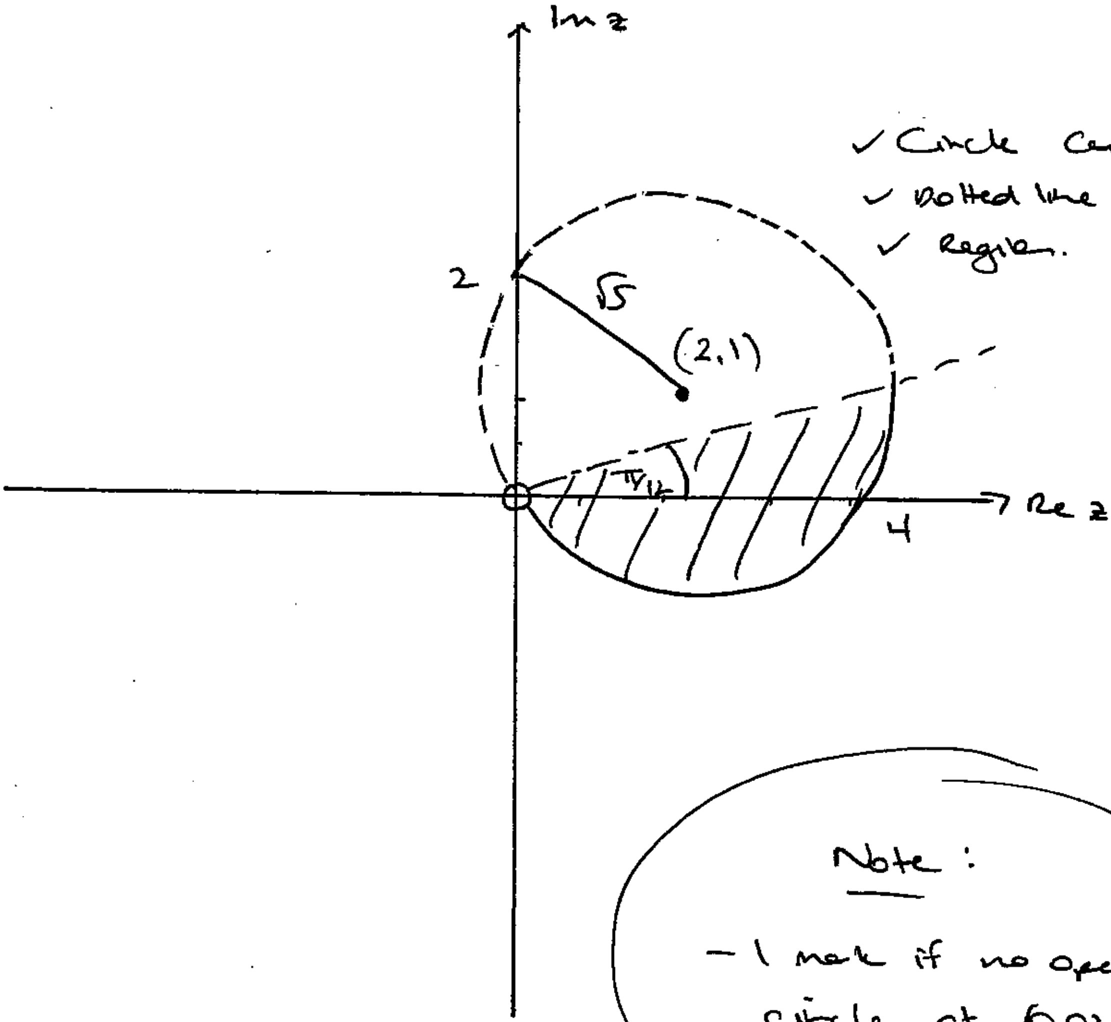
$(1-3\omega+\omega^2)(1+\omega-8\omega^2) = (-3\omega-\omega)(-\omega^2-8\omega^2)$

$= (-4\omega)(-9\omega^2)$

$= 36\omega^3$

$= 36.$

e)



(4)

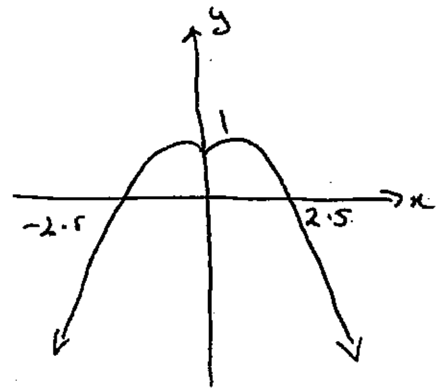
✓ Circle Centre (2,1)
✓ Dotted line arg $\pi/12$.
✓ Region.

Note:

−1 mark if no open circle at (0,0).

Section B

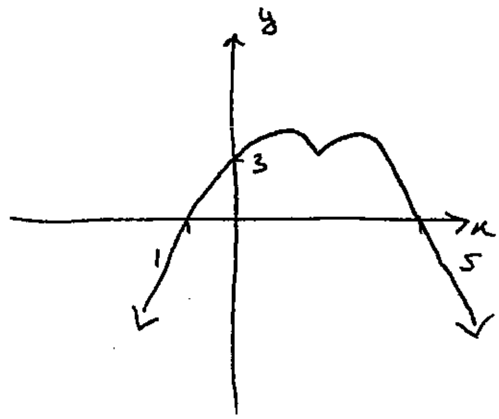
Q 2/ a) (i)



$y = f(x) - 2$

✓ shape + y intercept.

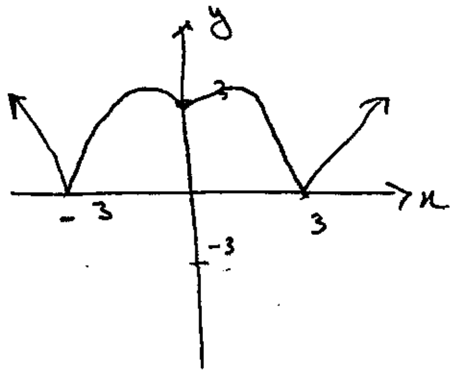
(ii)



$y = f(x-2)$

✓ shape + x intercepts/y-int.

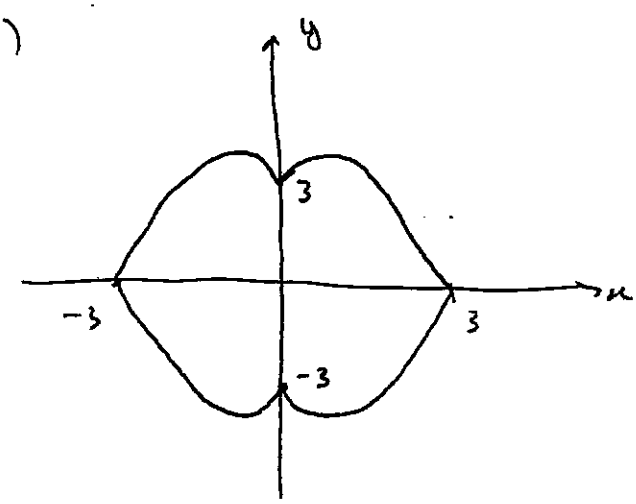
(iii)



$y = |f(x)|$

✓ shape

(iv)



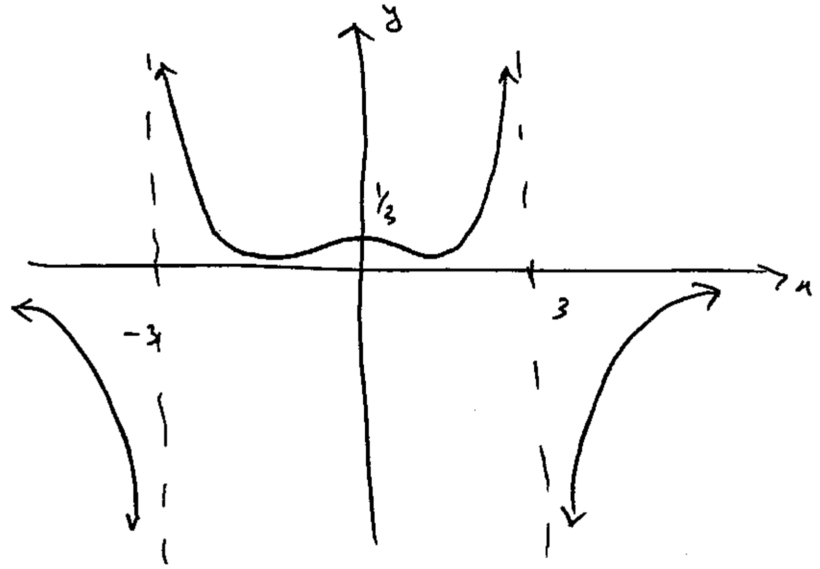
$y^2 = f(x)$

✓ shape

✓ x/y int.

(6)

2a) (v) $y = \frac{1}{f(x)}$



shape ✓
y-int ✓

Q2(b) (i) $f(x) = \dfrac{e^x - 1}{e^x + 1}$

$$f(-x) = \frac{e^{-x} - 1}{e^{-x} + 1} \qquad -f(x) = -\left(\frac{e^x - 1}{e^x + 1}\right)$$

$$= \frac{\frac{1}{e^x} - 1}{\frac{1}{e^x} + 1} \qquad = \frac{1 - e^x}{1 + e^x} \quad \checkmark$$

$$= \frac{1 - e^x}{1 + e^x}$$

$f(-x) = -f(x)$ hence odd.

(ii) $$f'(x) = \frac{(e^x + 1) \cdot e^x - (e^x - 1) e^x}{(e^x + 1)^2}$$

$$= \frac{\cancel{e^{2x}} + e^x - \cancel{e^{2x}} + e^x}{(e^x + 1)^2} \quad \checkmark$$

$$= \frac{2e^x}{(e^x + 1)^2} \quad \checkmark$$

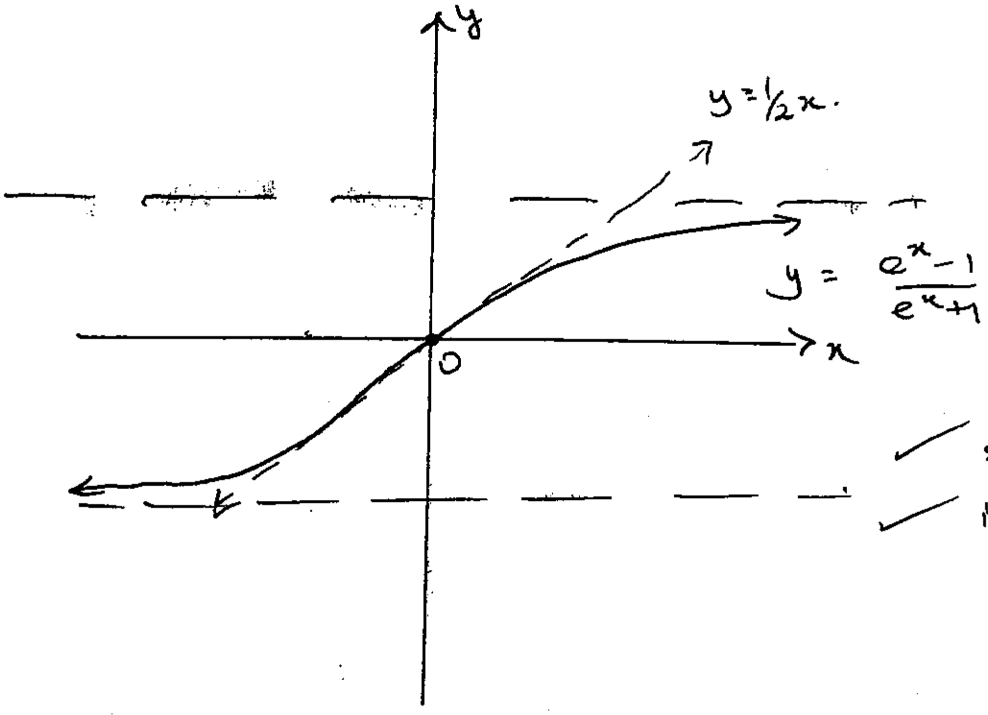
$f'(x) > 0$ for all values of x.

Hence monotonically increasing.

(iii) $$f'(0) = \frac{2e^{(0)}}{(e^0 + 1)^2} \quad \checkmark$$

$$= \frac{1}{2} \quad \checkmark$$

(iv) Over pg



✓ shape

✓ Intercept

as $x \to \infty$, $\dfrac{e^x - 1}{e^x + 1} \to 1$

as $x \to -\infty$, $\dfrac{e^x - 1}{e^x + 1} \to -1$ $\left[\text{i.e. } \dfrac{\frac{1}{e^x} - 1}{\frac{1}{e^x} + 1} \right]$

Both $\to 0$

(iv) $\dfrac{e^x - 1}{e^x + 1} = Mx$ has 3 solutions (real)

then line $y = Mx$ intersects $f(x)$ at 3 points.

Line must have gradient <u>between</u> $M = 0$ and $M = \frac{1}{2}$.

$$\boxed{0 < M < \tfrac{1}{2}.}$$

✓